CERTAIN EXACT SOLUTIONS TO STEADY-STATE PROBLEMS IN THE THEORY OF HEAT CONDUCTION APPLIED TO INHOMOGENEOUS BODIES

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A method is shown of constructing exact analytical solutions to steady-state problems in the theory of heat conduction where the thermal conductivity is a special kind of function of the space coordinates.

We consider the equation

$$\operatorname{div}(\lambda \operatorname{grad} T) = -f. \tag{1}$$

In place of T(x, y, z) we will introduce a new unknown function u(x, y, z) based on the transformation

$$u = \sqrt{\lambda}(T - T_0),\ T_0 = \text{const}. \tag{2}$$

For this function we have the equation

$$\Delta u - Mu = -\frac{f}{\sqrt{\lambda}}, \tag{3}$$

whose coefficient M(x, y, z) is determined from a given function λ(x, y, z) according to the equation

$$\Delta \sqrt{\lambda} - M\sqrt{\lambda} = 0. \tag{4}$$

We propose to select function M(x, y, z) so as to convert Eq. (3) into any well known equation of mathematical physics. Functions λ(x, y, z) which characterize the inhomogeneity of bodies cannot be arbitrary here, but they must belong to a certain class defined by Eq. (4).

We will consider only the simplest cases: $M \equiv 0$ and $M \equiv c$ (c = const).

1. $M \equiv 0$. For this case we have equations

$$\Delta u = -\frac{f}{\sqrt{\lambda}}, \tag{5}$$

$$\Delta \sqrt{\lambda} = 0. \tag{6}$$

If $\sqrt{\lambda(x, y, z)}$ is a harmonic function, therefore, then the steady-state problems in the theory of heat conduction reduce to boundary-value problems for either the Poisson equation or, when f = 0, the Laplace equation.

Example. To determine the steady-state temperature distribution in an infinitely long beam whose rectangular cross section is defined by segments of the four straight lines x = 0, x = a, y = 0, y = b under the following conditions: three sides of the beam are at temperature T_0, while a given temperature distribution T(x, b) = F(x) is maintained on the fourth side, and there are no heat sources (f = 0); λ is a function of coordinates x, y on a beam cross section and $\sqrt{\lambda(x, y)}$ is a harmonic function.

It is easy to determine function u(x, y) by the Fourier method. Then, according to (2), we obtain a solution to the problem in the form

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$$T(x, y) = T_0 + \frac{1}{\sqrt{\lambda(x, y)}} \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{a} \operatorname{sh} \frac{n\pi y}{a}, \tag{7}$$

where

$$B_n = \frac{2}{a \operatorname{sh}(n\pi b/a)} \int_0^a \sqrt{\lambda(\xi, b)}\,[F(\xi) - T_0] \sin \frac{n\pi \xi}{a}\, d\xi. \tag{8}$$

2. $M \equiv c$, $c = \text{const}$. Equations (3) and (4) yield

$$\Delta u - cu = -\frac{f}{\sqrt{\lambda}}, \tag{9}$$

$$\Delta \sqrt{\lambda} - c\sqrt{\lambda} = 0. \tag{10}$$

If function $\lambda(x, y, z)$ satisfies Eq. (10), then the steady-state problems in the theory of heat conduction reduce to boundary-value problems for Eq. (9). Useful analytical solutions to boundary-value problems for Eq. (9) can in a few cases be obtained by conventional methods.

Example. To determine the steady-state temperature distribution in an infinitely long beam whose rectangular cross section is defined by segments of the four straight lines $x = 0$, $x = a$, $y = 0$, $y = b$ under the following conditions: all sides are at the same temperature T_0 and $f(x, y)$, $\lambda(x, y)$ are known functions of the space coordinates in a beam cross section, where function $\lambda(x, y)$ satisfies Eq. (10).

The solution to the corresponding Dirichlet problem for Eq. (9) is obtained in the form of a binary trigonometric series [1]. After simple transformations and application of formula (2), this solution yields

$$T(x, y) = T_0 + \frac{1}{\sqrt{\lambda(x, y)}} \int_0^a \int_0^b \frac{f(\xi, \eta)}{\sqrt{\lambda(\xi, \eta)}} G(x, y; \xi, \eta)\, d\xi d\eta, \tag{11}$$

where

$$G(x, y; \xi, \eta) = \frac{4}{ab} \sum_{m, n=1}^{\infty} \frac{\sin(m\pi x/a)\sin(n\pi y/b)\sin(m\pi\xi/a)\sin(n\pi\eta/b)}{(m\pi/a)^2 + (n\pi/b)^2 + c} \tag{12}$$

is a Green function.

NOTATION

x, y, z	are the rectangular Cartesian coordinates;
$T(x, y, z)$	is the temperature;
$\lambda(x, y, z)$	is the thermal conductivity;
$f(x, y, z)$	is the intensity of heat sources;
∇	is the Laplace operator.

LITERATURE CITED

1. L. V. Kantorovich and V. I. Krylov, Approximation Methods in Higher Analysis [in Russian], Fizmatgiz, Moscow—Leningrad (1962), Ed. 5.